ISRAEL JOURNAL OF MATHEMATICS 102 (1997), 179–187

POLYNOMIALLY CONTINUOUS OPERATORS*

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ABSTRACT

A mapping between Banach spaces is said to be polynomially continuous if its restriction to any bounded set is uniformly continuous for the weak polynomial topology. Every compact (linear) operator is polynomially continuous. We prove that every polynomially continuous operator is weakly compact.

Throughout, X and Y are Banach spaces, S_X the unit sphere of X, and $\mathbf{N}$ stands for the natural numbers. Given $k \in \mathbf{N}$, we denote by $\mathcal{P}(^kX)$ the space of all k-homogeneous (continuous) polynomials from X into the scalar field $\mathbf{K}$ (real or complex). We identify $\mathcal{P}(^0X) = \mathbf{K}$, and denote $\mathcal{P}(X) := \sum_{k=0}^{\infty} \mathcal{P}(^kX)$. For the general theory of polynomials on Banach spaces, we refer to [11]. As usual, e_n stands for the sequence $(0, \ldots, 0, 1, 0, \ldots)$ with 1 in the nth position.

* Supported in part by DGICYT Project PB 93–0452 (Spain).
Received March 31, 1996

To each polynomial $P \in \mathcal{P}(^kX)$ we can associate a unique symmetric k-linear mapping $\hat{P}$: $X \times \overset{(k)}{\ldots} \times X \to \mathbf{K}$ so that $P(x) = \hat{P}(x, \ldots, x)$ for all $x \in X$. It is well known (see [11, Theorem 2.2]) that

$$\|P\| \le \|\hat{P}\| \le \frac{k^k}{k!}\|P\|.$$

Following [2], we say that a mapping f: $X \to Y$ is **polynomially continuous** (**P-continuous**, for short) if, for every $\epsilon > 0$ and bounded $B \subset X$, there are a finite set $\{P_1, \ldots, P_n\} \subset \mathcal{P}(X)$ and $\delta > 0$ so that $\|f(x) - f(y)\| < \epsilon$ whenever $x, y \in B$ satisfy $|P_j(x-y)| < \delta$ $(1 \le j \le n)$.

Clearly, the definition may be restated assuming that the polynomials $P_1, \ldots, P_n$ are homogeneous.

Suppose we require the polynomials $\{P_1, \ldots, P_n\} \subset \mathcal{P}(X)$ in the above definition to be of degree one, i.e., to be continuous linear forms on X. Then we obtain that f is weakly uniformly continuous on bounded subsets, a notion that has been studied by many authors (see [2]). Since a (linear) operator is compact if and only if it is weakly (uniformly) continuous on bounded sets [4, Proposition 2.5], every compact operator is P-continuous.

We say that a net $(x_\alpha) \subset X$ converges to x in the **weak polynomial topology** (*pw*-**topology,** for short) [5, §6] if for every $P \in \mathcal{P}(X)$ we have $P(x_\alpha) \to P(x)$. Taking advantage of an idea of [1, Proposition 4], we now show that the *pw* topology is semilinear [6, p. 265]. First, note that for $P \in \mathcal{P}(^kX)$ and $x, y \in X$,

$$\begin{aligned} P(x+y) &= \hat{P}(x+y, \overset{(k)}{\ldots}, x+y) \\ &= \sum_{j=0}^{k} \binom{k}{j} \hat{P}\left(x, \overset{(j)}{\ldots}, x, y, \overset{(k-j)}{\ldots}, y\right) \\ &=: \sum_{j=0}^{k} \binom{k}{j} \hat{P}\left(x^j, y^{k-j}\right). \end{aligned}$$

PROPOSITION 1: *A net $(x_\alpha) \subset X$ is pw-convergent to x if and only if $P(x_\alpha - x) \to 0$ for every $P \in \mathcal{P}(X)$.*

Proof: Clearly, it is enough to prove the result for homogeneous polynomials.

Suppose $P(x_\alpha - x) \to 0$ for all $P \in \mathcal{P}(X)$. Given $P \in \mathcal{P}(^kX)$, we have

$$\begin{aligned} P(x_\alpha) - P(x) &= P(x + x_\alpha - x) - P(x) \\ &= \sum_{j=0}^{k-1} \binom{k}{j} \hat{P}\left(x^j, (x_\alpha - x)^{k-j}\right) \\ &\to 0, \end{aligned}$$

since $\hat{P}\left(x^j, \cdot\right) \in \mathcal{P}(^{k-j}E)$. Therefore, (x_α) is pw-convergent to x.

Conversely, suppose (x_α) is pw-convergent to x. Choose $P \in \mathcal{P}(^kX)$. Then

$$\begin{aligned} P(-x + x_\alpha) &= \sum_{j=0}^{k} \binom{k}{j} \hat{P}\left((-x)^j, x_\alpha^{k-j}\right) \\ &\to \sum_{j=0}^{k} \binom{k}{j} \hat{P}\left((-x)^j, x^{k-j}\right) \\ &= P(x - x) \\ &= 0, \end{aligned}$$

since $\hat{P}\left((-x)^j, \cdot\right) \in \mathcal{P}(^{k-j}X)$. ∎

We shall use this result without explicit mention.

It is clear that a mapping $f\colon X \to Y$ is pw-continuous on bounded sets if and only if for every $x \in X$, $\epsilon > 0$ and bounded $B \subset X$ with $x \in B$, there are $\delta > 0$ and $\{P_1, \ldots, P_n\} \subset \mathcal{P}(X)$ so that we have $\|f(x) - f(y)\| < \epsilon$ whenever $|P_j(x - y)| < \delta$ for $1 \le j \le n$ and $y \in B$. Therefore, an operator is P-continuous if and only if it is pw-continuous on bounded sets.

Not every weakly compact operator is P-continuous. Indeed, if $X = T^*$, the original Tsirelson space, take a weakly null sequence $(x_n) \subset S_X$. Then, (x_n) is pw-null [1]. Therefore, the identity map on X is not P-continuous.

The following Lemma, whose proof is given later on, plays a key role in our main result:

LEMMA 2: *Given $P \in \mathcal{P}(^k\ell_1)$ with k even, and $\epsilon > 0$, there are $N \in \mathbf{N}$ and $t \in S_{\ell_1}$, so that $|P(t)| < \epsilon$ and*

$$t = \frac{1}{2N}\left(e_{p_1} + \cdots + e_{p_N} - e_{p_{N+1}} - \cdots - e_{p_{2N}}\right),$$

where $p_1 < \cdots < p_{2N}$.

Using it, we can prove the main theorem:

THEOREM 3: *Every P-continuous operator is weakly compact.*

Proof: Let $T: X \to Y$ be a P-continuous operator, and assume it is not weakly compact. We can find operators $U: \ell_1 \to X$, $S: \ell_1 \to \ell_\infty$ and $V: Y \to \ell_\infty$, with $S((t_n)) := \left(\sum_{i=1}^n t_i\right)_n$, so that $VTU = S$ (see [10, Theorem 8.1]). Then S must be P-continuous.

There is a pw-null net in S_{ℓ_1}, with elements of the form

$$t = \frac{1}{2N}\left(e_{p_1} + \cdots + e_{p_N} - e_{p_{N+1}} - \cdots - e_{p_{2N}}\right). \tag{1}$$

Indeed, given a finite set of homogeneous polynomials $\{P_1, \ldots, P_n\} \subset \mathcal{P}(\ell_1)$, if ℓ_1 is constructed over the real field, we set $P := P_1^{\alpha_1} + \cdots + P_n^{\alpha_n}$ so that P is a homogeneous polynomial of even degree. By Lemma 2, given $\epsilon > 0$, there is $t \in S_{\ell_1}$ of the form (1) so that $|P(t)| < \epsilon$.

Since $\|S(t)\| = 1/2$, S is not pw-continuous on the unit ball, a contradiction.

If ℓ_1 were complex, we would need an easy adaptation of Lemma 2 for a finite set of polynomials that may be assumed of even degree. ∎

In order to prove Lemma 2, we shall give a description of the polynomials on ℓ_1 in the spirit of Ryan's paper [12]. Following his notation, we write $\mathbf{N}_k^{(\mathbf{N})}$ for the set of *multi-indices of degree k*, i.e., the set of sequences $m = (m_j)_{j=1}^\infty$, with $m_j \in \mathbf{N}$ and $\sum_{j=1}^\infty m_j = k$. We let $m! = \prod_{j=1}^\infty m_j!$, where the usual convention $0! = 1$ is observed. If $a = (a_j)$ is a sequence of scalars, then $a^m := \prod_{j=1}^\infty a_j^{m_j}$, where 0^0 is defined to be 1.

LEMMA 4: *Every $P \in \mathcal{P}(^k\ell_1)$ may be written in the form $P(t) = \sum_{m \in \mathbf{N}_k^{(\mathbf{N})}} a_m t^m$ for $t \in \ell_1$, with scalar coefficients a_m satisfying the estimate*

$$|a_m| \frac{m^m}{k^k} \le C_k \|P\|,$$

for some constant $C_k > 0$ depending on k. If ℓ_1 is complex, we may take $C_k = 1$, and in the real case, $C_k = (2k)^k/k!$.

Proof: Let $P \in \mathcal{P}(^k\ell_1)$. If $t = (t_i) \in \ell_1$ has bounded support, we have

$$\begin{aligned} P(t) &= \hat{P}\left(\sum_{i=1}^\infty t_i e_i, \overset{(k)}{\ldots}, \sum_{i=1}^\infty t_i e_i\right) \\ &= \sum_{m \in \mathbf{N}_k^{(\mathbf{N})}} \frac{k!}{m!} \hat{P}\left(e_1, \overset{(m_1)}{\ldots}, e_1, \ldots, e_j \overset{(m_j)}{\ldots}, e_j, \ldots\right) t^m. \end{aligned}$$

For each $m = (m_j) \in \mathbf{N}_k^{(\mathbf{N})}$, let

$$a_m := \frac{k!}{m!}\hat{P}\left(e_1, \overset{(m_1)}{\ldots}, e_1, \ldots, e_j \overset{(m_j)}{\ldots}, e_j, \ldots\right).$$

If ℓ_1 is complex, the Harris inequalities [7] (see [3, Corollary 8]) state that:

$$\left|\hat{P}\left(e_1, \overset{(m_1)}{\ldots}, e_1, \ldots, e_j \overset{(m_j)}{\ldots}, e_j, \ldots\right)\right| \leq \frac{m!}{m^m}\frac{k^k}{k!}\|P\|.$$

Therefore,

$$|a_m|\frac{m^m}{k^k} \leq \|P\|.$$

Now, suppose ℓ_1 is real. Let $\ell_1^c = \ell_1 + i\ell_1$ with the ℓ_1-norm, and define the complexification $Q \in \mathcal{P}(^k\ell_1^c)$ of P by

$$Q(x+iy) = \sum_{j=0}^{k} \binom{k}{j} i^j \hat{P}\left(x^{k-j}, y^j\right).$$

By the above, if x, y have bounded support, we may write

$$Q(x+iy) = \sum_{m \in \mathbf{N}_k^{(\mathbf{N})}} a_m (x+iy)^m.$$

In particular, for $y = 0$,

$$P(x) = \sum_{m \in \mathbf{N}_k^{(\mathbf{N})}} a_m x^m,$$

and

$$|a_m|\frac{m^m}{k^k} \leq \|Q\| \leq \sum_{j=0}^{k} \binom{k}{j} \|\hat{P}\| \leq \frac{(2k)^k}{k!}\|P\|.$$

The series $\sum_{m \in \mathbf{N}_k^{(\mathbf{N})}} a_m t^m$ defines a k-homogeneous polynomial on ℓ_1. Indeed, since

$$\frac{m!}{k!} \leq \frac{e^k m^m}{k^k} \quad \text{for all } m \in \mathbf{N}_k^{(\mathbf{N})}$$

(see [12, Lemma 3.2]), we have

$$\begin{aligned}\left|\sum_{m \in \mathbf{N}_k^{(\mathbf{N})}} a_m t^m\right| &\leq \sum_{m \in \mathbf{N}_k^{(\mathbf{N})}} |a_m||t|^m \\ &= \sum_{m \in \mathbf{N}_k^{(\mathbf{N})}} |a_m|\frac{m!}{k!}\frac{k!}{m!}|t|^m \\ &\leq C_k e^k \|P\|\|t\|^k,\end{aligned}$$

where we have used the multinomial theorem, stating that for $t = (t_i) \in \ell_1$, we have

$$\left(\sum_{i=1}^{\infty} t_i\right)^k = \sum_{m\in \mathbf{N}_k^{(\mathbf{N})}} \frac{k!}{m!} t^m.$$

We have $P(t) = \sum_{m\in \mathbf{N}_k^{(\mathbf{N})}} a_m t^m$ for every $t \in \ell_1$ with finitely many nonzero coordinates. By density, the equality holds for all $t \in \ell_1$, and the proof is complete. ∎

Using a result of [9], we may slightly improve the constant in the real case, taking

$$C_k := \frac{1}{2}\left[(3\sqrt{2}+4)^k + (3\sqrt{2}-4)^k\right].$$

In the proof of Lemma 2, we shall use Ramsey's Theorem:

THEOREM 5 ([8, Lemma 29.1]): *Given an infinite subset $A \subseteq \mathbf{N}$, and $k, n \in \mathbf{N}$, we denote by $[A]^k$ the set of all k-tuples $(i_1, \dots, i_k)$ in A such that $i_1 < \cdots < i_k$. Let $\{M_1, \dots, M_n\}$ be a partition of $[A]^k$ into n disjoint sets. Then there is an infinite subset $H \subseteq A$ such that $[H]^k \subseteq M_r$ for some $1 \le r \le n$.*

Proof of Lemma 2: Let $P(t) := \sum_{m\in \mathbf{N}_k^{(\mathbf{N})}} a_m t^m$, and assume $|a_m| \le 1$.

FIRST STEP: CONSTRUCTION OF t: Denote $m(i_1, \dots, i_k) := e_{i_1} + \cdots + e_{i_k}$. We partition the closed unit disk of $\mathbf{K}$ into four disjoint subsets $D_1^1, \dots, D_1^4$ of diameter not greater than $\sqrt{2}$. With the notation of Theorem 5, let

$$M_1^r := \left\{(i_1, \dots, i_k) \in [\mathbf{N}]^k \colon a_{m(i_1,\dots,i_k)} \in D_1^r\right\} \qquad (1 \le r \le 4).$$

By Theorem 5, there is an infinite set $H_1 \subseteq \mathbf{N}$ so that $[H_1]^k \subset M_1^r$ for some $1 \le r \le 4$. To fix notation, assume $r = 1$. Divide M_1^1 into disjoint subsets $D_2^1, \dots, D_2^4$ of diameter not greater than $\sqrt{2}/2$. Let

$$M_2^r := \left\{(i_1, \dots, i_k) \in [H_1]^k \colon a_{m(i_1,\dots,i_k)} \in D_2^r\right\} \qquad (1 \le r \le 4).$$

Repeating the process, we obtain a sequence $H_1 \supseteq H_2 \supseteq \cdots$ of infinite subsets of $\mathbf{N}$ such that whenever $(i_1, \dots, i_k), (j_1, \dots, j_k) \in [H_n]^k$, we have

$$\left|a_{m(i_1,\dots,i_k)} - a_{m(j_1,\dots,j_k)}\right| \le 2^{-n+3/2}.$$

Now, choose $n \in \mathbf{N}$ with $2^{-n+3/2} < \epsilon$. Take $N > k$ large (see below), and $(p_1, \dots, p_{2N}) \in [H_n]^{2N}$. Let

$$t := \frac{1}{2N}\left(e_{p_1} + \cdots + e_{p_N} - e_{p_{N+1}} - \cdots - e_{p_{2N}}\right).$$

SECOND STEP: COMPUTING $P(t)$: Let $m \in \mathbf{N}_k^{(N)}$. We can assume that the support of m is contained in that of t, in which case we have $t^m = \pm 2^{-k}N^{-k}$. Otherwise, we would have $t^m = 0$.

Suppose first that $\|m\|_\infty > 1$. The number of m's that may be chosen with this condition is less than or equal to

$$\rho_1(N) := A_{2N}^1 + A_{2N}^2 + \cdots + A_{2N}^{k-1},$$

where A_{2N}^i is the number of arrangements of size i of $2N$ distinct elements. Hence

$$\left| \sum_{\|m\|_\infty > 1} a_m t^m \right| \leq \frac{1}{2^k N^k} \rho_1(N) < \frac{\epsilon}{3}$$

for N large enough.

Now, if $\|m\|_\infty = 1$, denoting by C_N^i the number of combinations of size i from N distinct elements, we have

$$A^+ := C_N^k + C_N^2 C_N^{k-2} + \cdots + C_N^{k-2} C_N^2 + C_N^k$$

multi-indices m with $t^m = 2^{-k}N^{-k}$, and

$$A^- := C_N^1 C_N^{k-1} + C_N^3 C_N^{k-3} + \cdots + C_N^{k-1} C_N^1$$

multi-indices m with $t^m = -2^{-k}N^{-k}$. The coefficient of N^k in A^+ is

$$\frac{1}{k!}\left[\binom{k}{0} + \binom{k}{2} + \cdots + \binom{k}{k}\right] = \frac{2^{k-1}}{k!}$$

and in A^-:

$$\frac{1}{k!}\left[\binom{k}{1} + \binom{k}{3} + \cdots + \binom{k}{k-1}\right] = \frac{2^{k-1}}{k!}.$$

If $A := \min\{A^+, A^-\}$, we can take sets

$$\mathcal{S}, \mathcal{T} \subset \left\{ m \in \mathbf{N}_k^{(N)} \colon \|m\|_\infty = 1 \right\},$$

both of cardinal A, so that we have $t^m > 0$ whenever $m \in \mathcal{S}$, while $t^m < 0$ if $m \in \mathcal{T}$. By the construction of t, we know that $|a_m - a_{m'}| < \epsilon$ for all $m \in \mathcal{S}$ and $m' \in \mathcal{T}$, and so

$$\left| \sum_{m \in \mathcal{S} \cup \mathcal{T}} a_m t^m \right| < \frac{\epsilon A}{2^k N^k} < \frac{\epsilon}{3},$$

for N large enough.

The set $\mathcal{R}$ of the remaining multi-indices, i.e., m's so that $\|m\|_\infty = 1$ and $m \notin \mathcal{S} \cup \mathcal{T}$, has a cardinal equal to the absolute value

$$\rho_2(N) := \left| \sum_{j=0}^{k} (-1)^j C_N^j C_N^{k-j} \right|$$

of a polynomial on N of degree $k-1$. Therefore,

$$\left| \sum_{m \in \mathcal{R}} a_m t^m \right| \leq \frac{1}{2^k N^k} \rho_2(N) < \frac{\epsilon}{3}$$

for N large enough. Hence

$$|P(t)| = \left| \sum_{\|m\|_\infty > 1} a_m t^m + \sum_{m \in \mathcal{S} \cup \mathcal{T}} a_m t^m + \sum_{m \in \mathcal{R}} a_m t^m \right| < \epsilon. \quad \blacksquare$$

ACKNOWLEDGEMENT: The authors wish to thank R. Gonzalo, J. A. Jaramillo and M. Lacruz for helpful discussions.

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